Commutative Property of Matrix Addition

$$A + B = B + A$$

Associative Property of Matrix Addition

$$(A + B) + C = A + (B + C)$$

EXAMPLE

Demonstrating the Commutative Property

$$
\begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 2 + (-1) & 3 + 2 & -1 + 1 \\ 4 + 5 & 0 + (-3) & 7 + 4 \end{bmatrix}
$$

$$
= \begin{bmatrix} -1 + 2 & 2 + 3 & 1 + (-1) \\ 5 + 4 & -3 + 0 & 4 + 7 \end{bmatrix}
$$

$$
= \begin{bmatrix} -1 & 2 & 1 \\ 5 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -1 \\ 4 & 0 & 7 \end{bmatrix}
$$

The Zero Matrix

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 2 by 2 square zero matrix

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 2 by 3 zero matrix

$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ 1 by 3 zero matrix

$$A + 0 = 0 + A = A$$

In other Words, the zero matrix is the additive identity in matrix algebra

Theorem

Matrix multiplication is not commutative.

2×3 3×1

Associative Property of Matrix Multiplication

3×1 2×3

$$A(BC) = (AB)C$$

Distributive Property

$$A(B + C) = AB + AC$$

Identity Property

If A is an m by n matrix, then

$$I_m A = A \quad \text{and} \quad A I_n = A$$

If A is an n by n square matrix, then

$$A I_n = I_n A = A$$

For an n by n square matrix, the entries located in row i, column i, $1 \leq i \leq n$, are called the **diagonal entries**. An n by n square matrix whose diagonal entries are 1's, while all other entries are 0's, is called the **identity matrix** $\boldsymbol{I_n}$. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and so on.

EXAMPLE

Multiplication with an Identity Matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Find: (a) AI_3 (b) I_2A (c) BI_2

2x3 2x3

$$\begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3+0+0 & 0+-2+0 & 0+0+1 \\ 0+0+0 & 0+4+0 & 0+0+-1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix}$$

EXAMPLE

Multiplication with an Identity Matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Find: (a) AI_3 (b) I_2A (c) BI_2

(c)

$$\begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2+0 & 0+4 \\ -1+0 & 0+3 \\ -3+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

EXAMPLE

Multiplication with an Identity Matrix

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 4 \\ -1 & 3 \\ -3 & 1 \end{bmatrix}$$

Find: (a) AI_3 (b) I_2A (c) BI_2

OBJECTIVE 4

4 Find the Inverse of a Matrix

DEFINITION

Let A be a square n by n matrix. If there exists an n by n matrix A^{-1}, read "A inverse," for which

$$AA^{-1} = A^{-1}A = I_n$$

then A^{-1} is called the **inverse** of the matrix A.

The determinant of A must not equal zero for there to be an inverse.

EXAMPLE

Multiplying a Matrix by Its Inverse

Show that the inverse of $A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ is A^{-1} $\begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$

$$\left[\begin{array}{cc|cc} -3 & -1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{array}\right] \xrightarrow{\frac{R_1}{-3} = R_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 4 & 2 & 0 & 1 \end{array}\right] \xrightarrow{-4R_1 + R_2 = R_2} \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{4}{3} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \cdot \frac{3}{2} = R_2} \left[\begin{array}{cc|cc} 1 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & 2 & \frac{3}{2} \end{array}\right] \xrightarrow{R_2 \cdot -\frac{1}{3} + R_1 = R_1} \left[\begin{array}{cc|cc} 1 & 0 & -1 & -\frac{1}{2} \\ 0 & 1 & 2 & \frac{3}{2} \end{array}\right]$$

EXAMPLE

Multiplying a Matrix by Its Inverse

Show that the inverse of $A = \begin{bmatrix} -3 & -1 \\ 4 & 2 \end{bmatrix}$ is $A^{-1} \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$

$-6 + 4$

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{-2} \cdot \begin{bmatrix} 2 & 1 \\ -4 & -3 \end{bmatrix}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \\ 2 & \frac{3}{2} \end{bmatrix}$$

EXAMPLE

Showing That a Matrix Has No Inverse

Show that the matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$ has no inverse.

$$\det = \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} = (4) - 4 = 0$$

Procedure for Finding the Inverse of a Nonsingular Matrix

To find the inverse of an n by n nonsingular matrix A, proceed as follows:

STEP 1: Form the matrix $[A|I_n]$.

STEP 2: Transform the matrix $[A|I_n]$ into reduced row echelon form.

STEP 3: The reduced row echelon form of $[A|I_n]$ will contain the identity matrix I_n on the left of the vertical bar; the n by n matrix on the right of the vertical bar is the inverse of A.